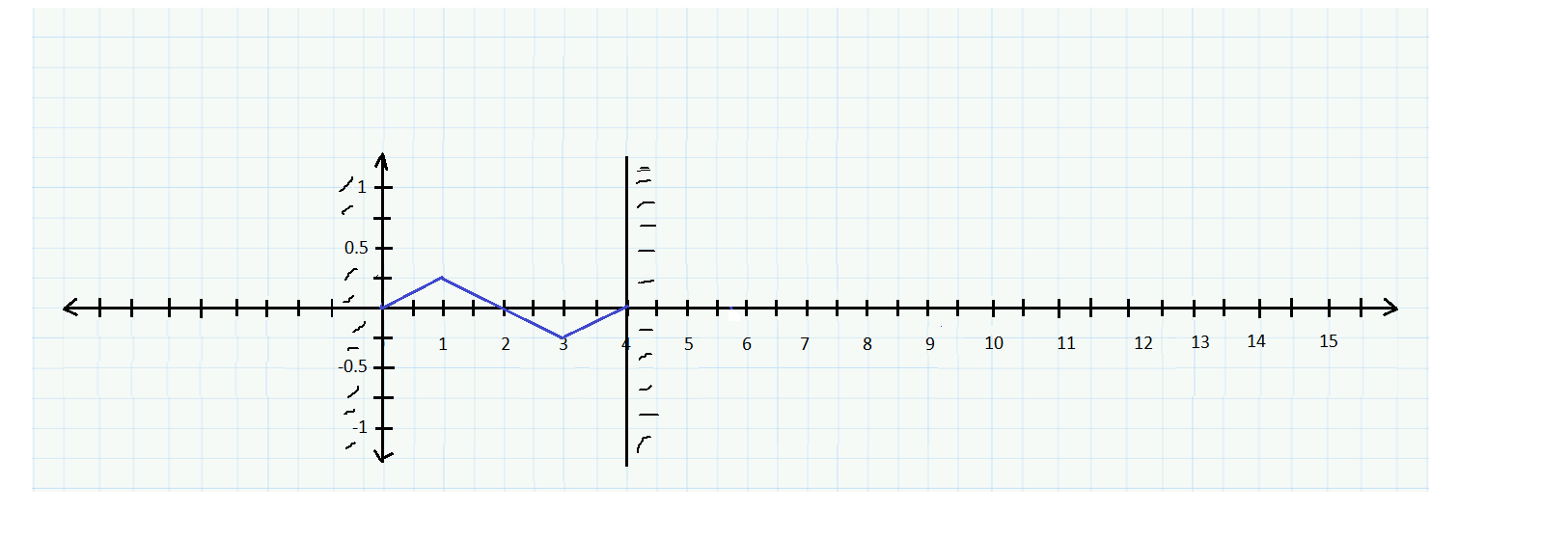
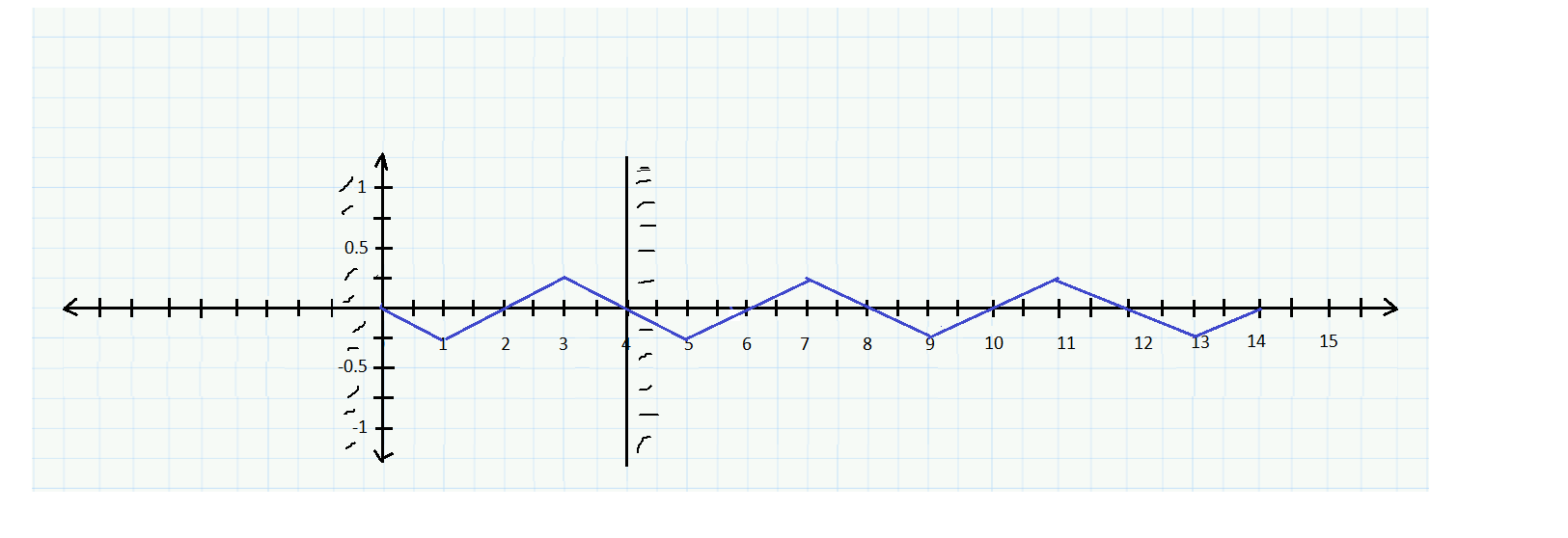
**Homework 5 Solutions**

**Problem 1.** I’d like to illustrate, to some degree, the difference between how a resonant wave will interfere with itself, and how a non-resonant wave will interfere with itself, when repeatedly reflecting off a boundary. A resonant wave should keep amplifying itself, whereas a non-resonant wave will not (not to say it will cancel itself out – it just won’t amplify itself). So to that end,

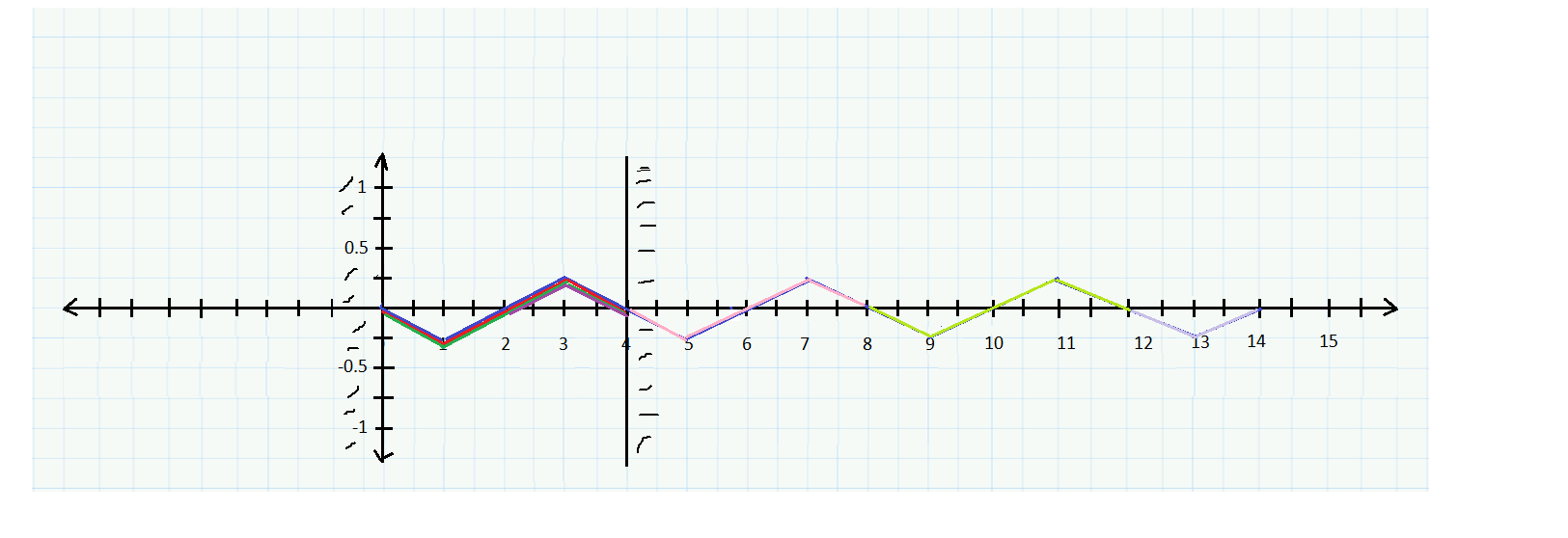
(a) Consider the resonant wave being generated below. Suppose it travels with a speed of 1 unit/second to the right, and will be reflected off the two infinitely hard boundaries at x = 0 and x = 4. What does the wave look like 10s later? In the three graphs below, draw the wave translated ‘across the boundary’ by 10s. Then draw the incident wave + the three reflected waves below that (there’s three because of repeated reflections off the walls). And then draw the superposition below that.



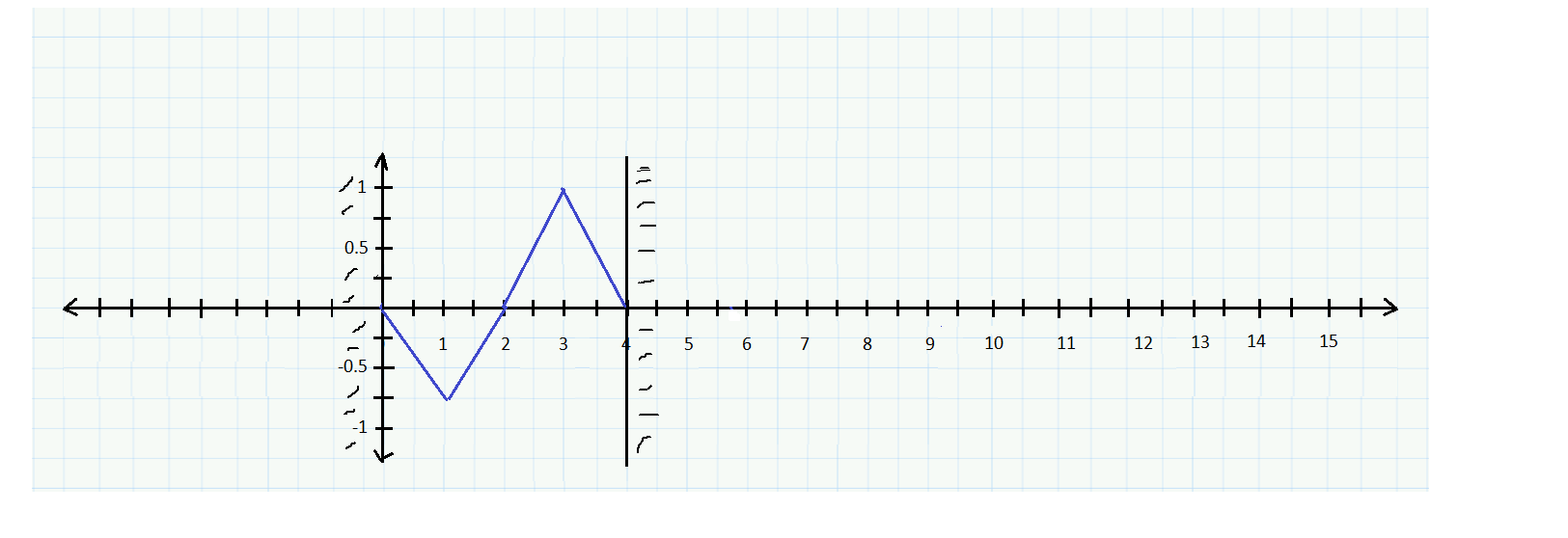
First we translate it forward 10s…



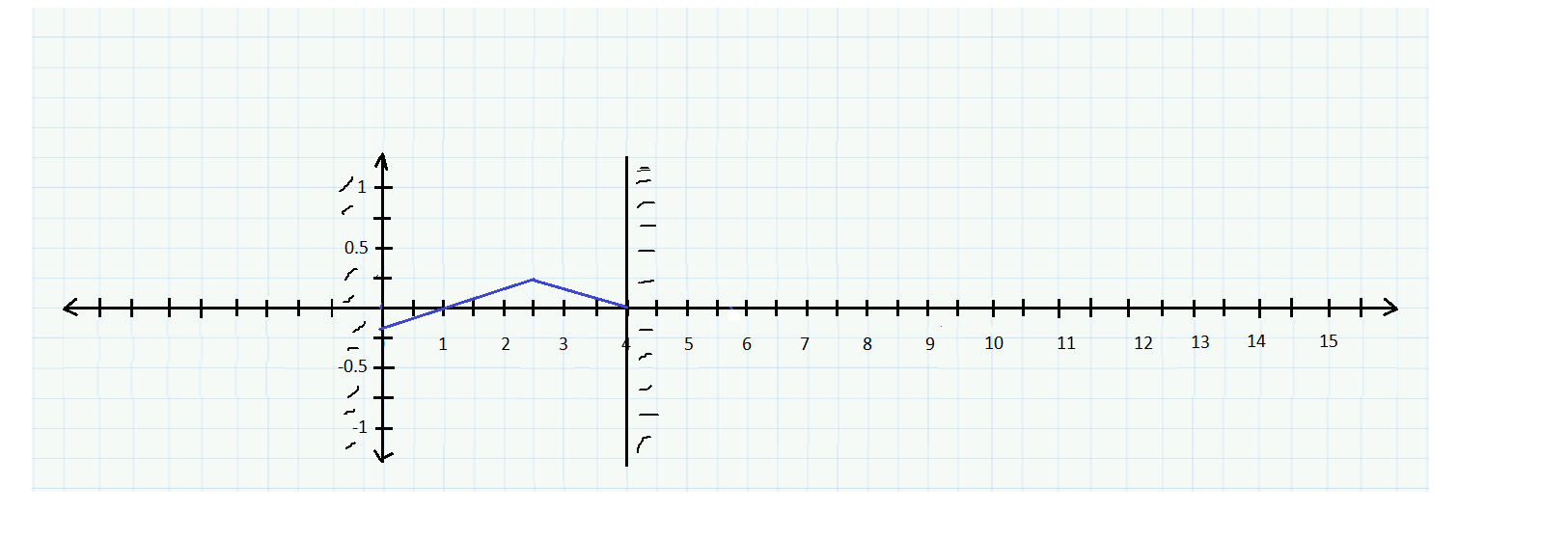
Then we have to do the reflections. The pink segment of the wave would reflect left-right and up-down once, and is drawn in red. The light green segment of the wave would reflect left-right twice and up-down twice, and is drawn in dark green. The light purple segment would reflect left-right and up-down thrice (I like that word), and is drawn in purple.



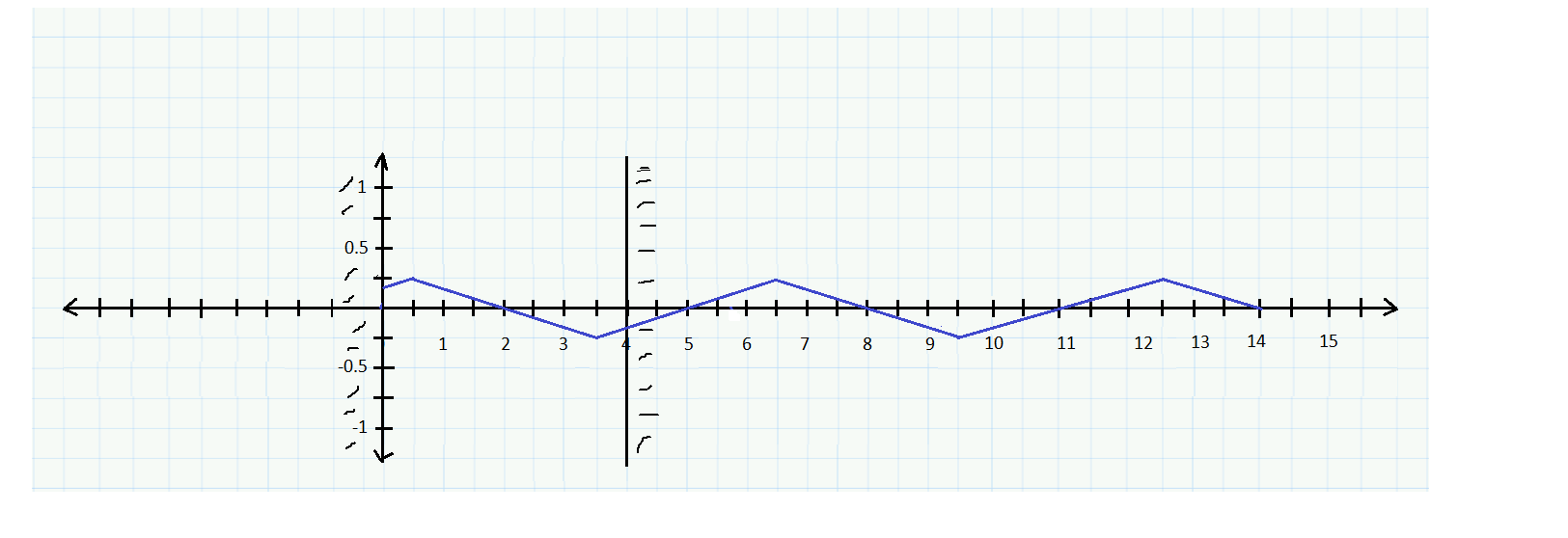
Superposition will look like:



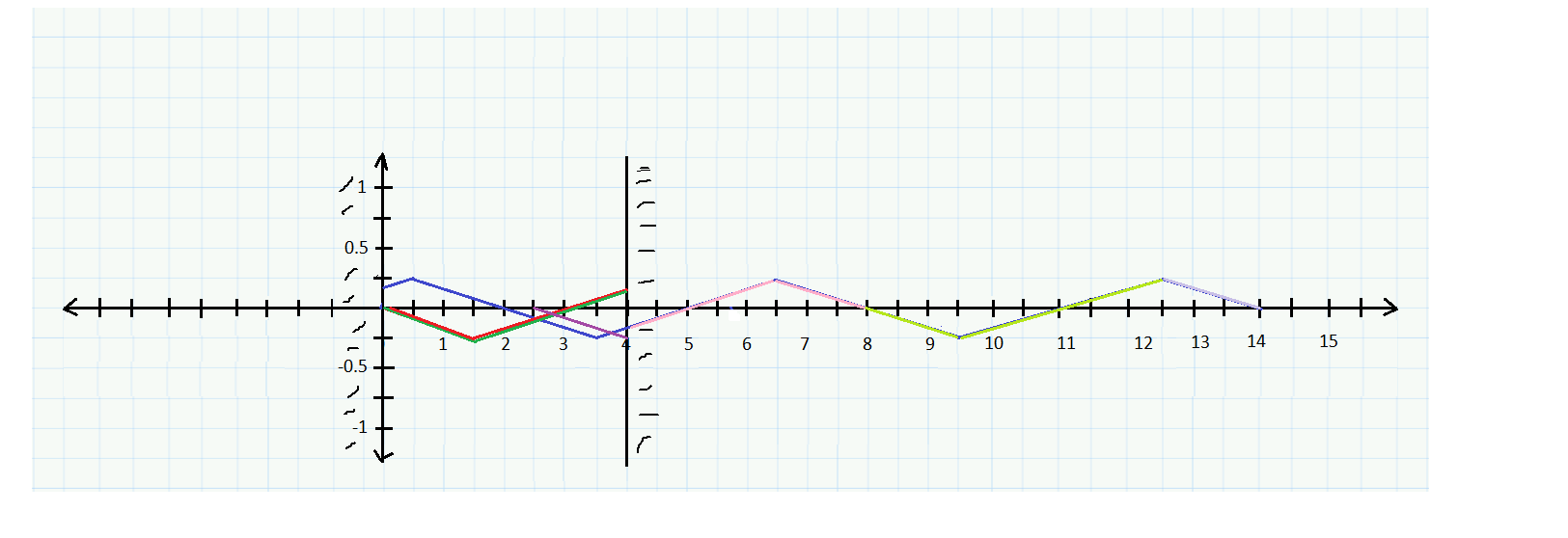
(b) Now consider a non-resonant wave being generated below. Let it also travel to the right at 1 unit/second, and be reflected off the two infinitely hard boundaries at x = 0 and x = 4. What does the wave look like 10s later? In the three graphs below, draw the wave translated ‘across the boundary’ by 10s. Then draw the incident wave + the three reflected waves below that. And then draw the superposition below that.



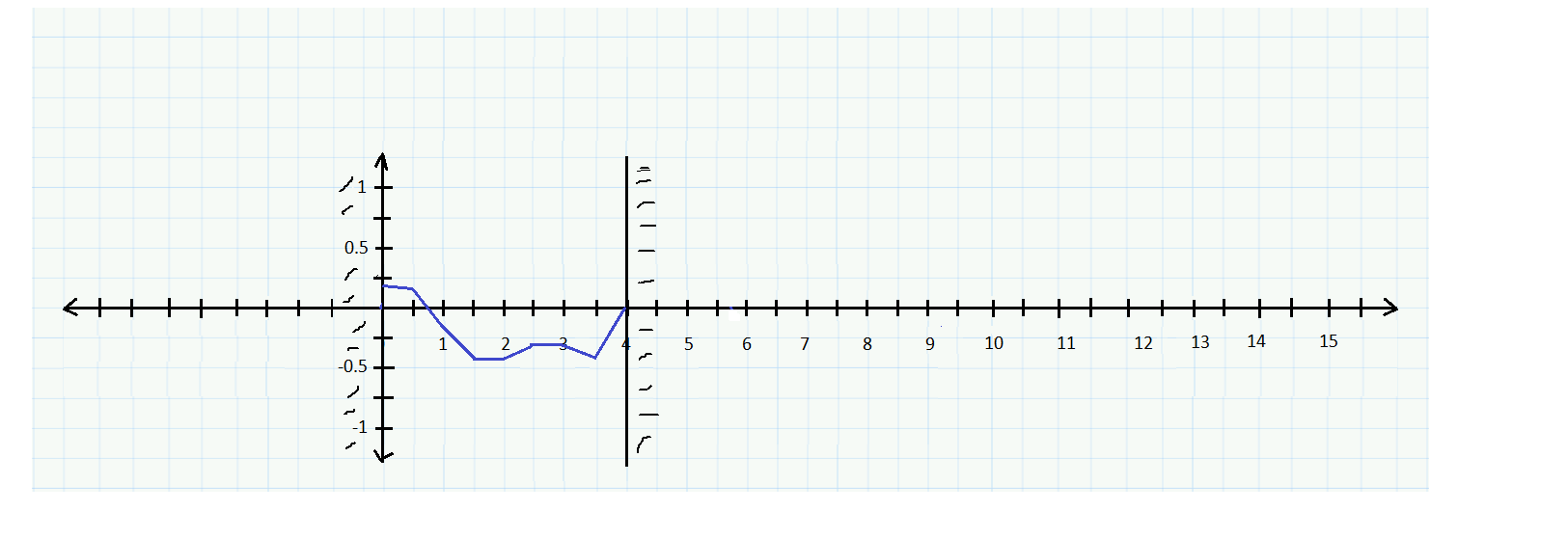
We translate it forward 10s…



Then we have to do the reflections. The pink segment of the wave would reflect left-right and up-down once, and is drawn in red. The light green segment of the wave would reflect left-right twice and up-down twice, and is drawn in dark green. The light purple segment would reflect left-right and up-down thrice (I like that word), and is drawn in purple. Observe the chaotic nature of the interference pattern.



Approximate superposition is drawn below:

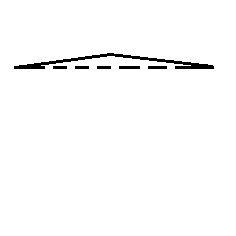


**Problem 2.** Musical aside: The notes that comprise the bottom octave on a piano are:

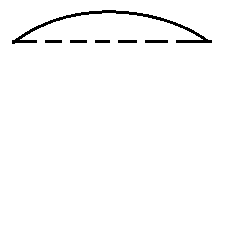
|  |  |
| --- | --- |
| **Note** | **Frequency (Hz)** |
| C1 | 32.7 |
| D1 | 36.7 |
| E1 | 41.2 |
| F1 | 43.7 |
| G1 | 49.0 |
| A1 | 55.0 |
| B1 | 61.7 |

Each successive octave doubles the frequencies. For instance Middle C, i.e. C4, has a frequency of C1∙23 = 262Hz. Now that that’s settled….

When you pluck a stringed instrument, you give it an initial shape D(x,0) that is somewhat triangular-looking. A triangular shape is not a sinusoidal shape, and so it will not be characterized by a *single* wavelength (and frequency) of oscillation like sinusoids are. Rather, the wavelength (and frequency) of oscillation will be comprised of certain percentages of the resonant waveforms’ wavelengths (and frequencies) of oscillation. The resonant waveform which will contribute most is the one that most closely resembles D(x,0). Since the initial shape of a plucked string looks like this:



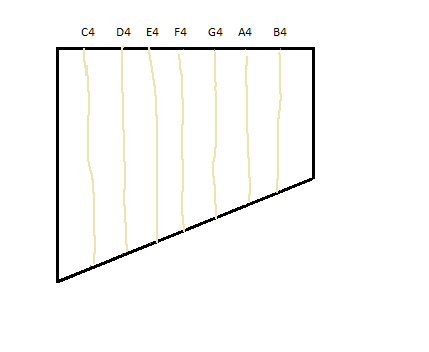
And the resonant waveform which most closely matches this shape is the ‘fundamental’, or ‘first’, one:



So the wavelength (frequency) that you hear the most when you pluck a string is the fundamental wavelength (frequency), though you will hear the higher resonances to a lesser extent, nonetheless.

(a) Now suppose you wanted to make a stringed instrument, like a harp. Supposing each string had a mass density μ = 0.010kg/m, and were tightened to a tension T = 400N, what ought to be the lengths of the seven strings so that their fundamental frequencies would be the following notes. Be sure to do the whole derivation starting from Δφ = 2πm, m1/2. ☺

|  |  |
| --- | --- |
| **Note** | **Length (cm)** |
| C4 |  |
| D4 |  |
| E4 |  |
| F4 |  |
| G4 |  |
| A4 |  |
| B4 |  |



Resonant wavelengths are:



And so the resonant frequencies are:



And so the fundamental frequency is:



And so the requisite lengths are:



So,

|  |  |  |
| --- | --- | --- |
| **Note** | **Frequency (Hz)** | **Length (cm)** |
| C4 | C1∙8 = 262 | 38.2 |
| D4 | D1∙8 = 294 | 34.0 |
| E4 | E1∙8 = 330 | 30.3 |
| F4 | F1∙8 = 350 | 28.7 |
| G4 | G1∙8 = 392 | 25.5 |
| A4 | A1∙8 = 440 | 22.7 |
| B4 | B1∙8 = 494 | 20.2 |

(b) Which harmonics of the string, if any, would correspond to the next two octaves? And how could these notes be played?

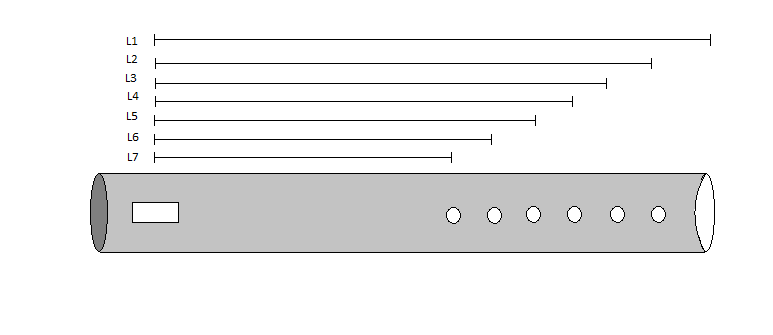
The second, third, and fourth harmonics are:



So the second corresponds to the next octave, and the fourth corresponds to the octave after that. You could play these by plucking the string in a manner similar to these harmonics: a figure eight for the second harmonic, and a figure ‘16’ for the fourth. Or better, just pluck half the the string for the second, and a quarter of the string for the fourth.

**Problem 3.** When you play a wind or reed instrument, like a flute, or clarinet you do not control the shape of the disturbance you create in the body of the instrument, like you do with a string instrument. Rather, by blowing over the opening (‘slow’, or ‘fast’, etc.) you control the *frequency(ies)* of disturbance you create in the instrument’s air column. The faster you blow (overblowing it’s called), the higher the frequency you may produce. Blowing over the opening will create waves with a *range* of frequencies actually; only the ones whose frequencies match a resonant waveform will be amplified.

(a) Now suppose you wanted to make a flute. To create the notes in the octave, we need to have seven different lengths of air columns. A shortcut is to simply drill 6 holes. By covering up all the holes, we have an open ended column of air L1 long. If we cover up all but the last, then we have an open-ended column L2 long, and so on. So what should these L’s be to create notes C4 through B4 again? Be sure to do the whole derivation starting from Δφ = 2πm, m1/2. Take the air to have molar mass mmol = 0.029kg, to be diatomic (γ = 1.4), and to be at temperature T = 310K.

****

The resonant wavelengths are:



And frequencies,



The fundamental frequencies are:



Solving for length we have:



Filling in the same frequencies as before, we get:

|  |  |
| --- | --- |
| **Note** | **Length (cm)** |
| C4 | 67.2 |
| D4 | 59.9 |
| E4 | 53.3 |
| F4 | 50.4 |
| G4 | 44.9 |
| A4 | 40.0 |
| B4 | 35.6 |

(b) Suppose the room were colder. What would you have to do to these lengths to compensate (this is why you can adjust the length of wind instrument)?

If T is lower, then v is lower. And we can see from L = 176/f, that we’d have to shorten L.

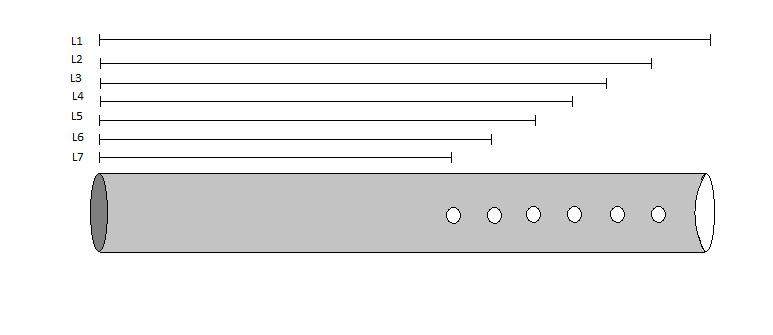
(c) Which harmonics of the flute, if any, would correspond to the next two octaves? And how could these notes be played in principle (kind of already said it)?

The second, third, and fourth harmonics are:



So the second corresponds to the next octave, and the fourth corresponds to the octave after that. You could play these by overblowing, for instance.

**Problem 4.** Suppose you wanted to make a recorder (like a clarinet), which differs from a flute in that the closed end over which you blow makes that end of the instrument a ‘hard’ or ‘closed’ end. We make a similar construction. So what should these L’s be to create notes C4 through B4 again? Be sure to do the whole derivation starting from Δφ = 2πm, m1/2, s’il vous plait. And take the air to have molar mass mmol = 0.029kg, to be diatomic (γ = 1.4), and to be at temperature T = 310K. Note the mouthpiece isn’t really shown, because I’m using MS Paint, and yeah.

****

The resonant wavelengths are:



And frequencies,



The fundamental frequencies are:



Solving for length we have:



Filling in the same frequencies as before, we get:

|  |  |
| --- | --- |
| **Note** | **Length (cm)** |
| C4 | 33.6 |
| D4 | 30.0 |
| E4 | 26.6 |
| F4 | 25.2 |
| G4 | 22.4 |
| A4 | 20.0 |
| B4 | 17.8 |

(b) Suppose the room were warmer. What would you have to do to these lengths to compensate (this is why you can adjust the length of such instruments too)?

If T is higher, then v is higher. And we can see from L = 88/f, that we’d have to lengthen L.

(c) Which harmonics of the recorder, if any, would correspond to the next two octaves?

The second, third, and fourth harmonics are:

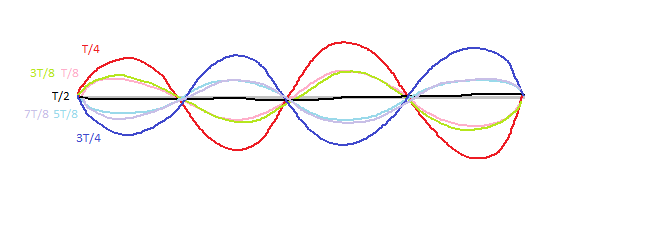


So actually, none of the higher harmonics corresponds to the octave up, or any of the higher octaves. So these higher notes couldn’t *simply* be played by overblowing. You’d have to do that in combination with pressing some more keys (closing holes, etc.).

**Problem 5.** We gonna draw some pictures! For each of the following three cases, and for the 4th resonant waveform, specify its speed, wavelength, frequency, and period. And then draw what it looks like in seven 1/8 period incrememts (should return to its initial state by the eighth increment), starting from the equilibrium position at t = 0.

(a) A harp string with length ℓ = 75cm, mass m = 0.007kg, under tension T = 600N.

Looks like,



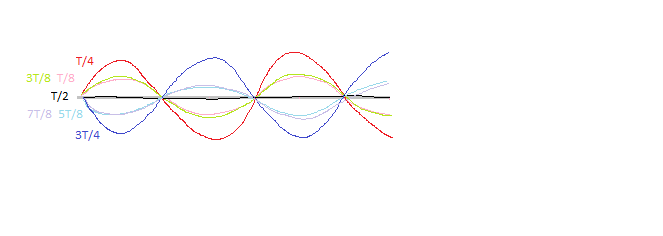


And:



(b) A trumpet with tube length ℓ = 50cm (all valves closed), enclosing air (γ = 1.4, molar mass m = 0.029kg, gas constant R = 8.31 J/mol∙K) at temperature T = 310K.

Looks like,



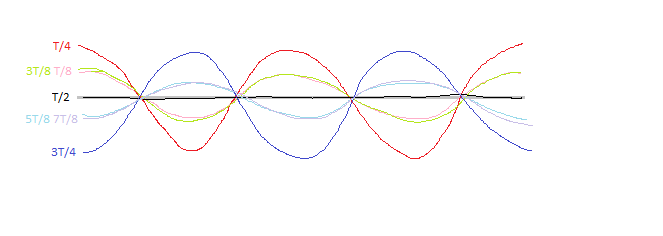


And,



(c) A flute of length ℓ = 30cm (all valves closed), enclosing air (γ = 1.4, molar mass m = 0.029kg, gas constant R = 8.31 J/mol∙K) at temperature T = 310K.

Looks like,





And,



**Problem 6.** The resonant waveleforms analyzed above were those of the string or of the air *inside* the instrument. But you don’t hear the string itself or the air inside the instrument itself. The soundwaves you hear are those which these resonant waveforms setup in the air *surrounding* the instrument. Suppose the surrounding air temperature is T = 280K (same γ, molar mass, R). In each of the cases above, give the velocity, wavelength, frequency, and period of the wave *you* hear.

(a) Easy peasy. We analyze this more or less like a wave transmitting into another medium. Speed and wavelength is determined by medium, but frequency (and period) is invariant.



(b) Same as above,

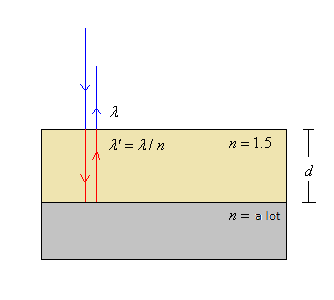


(c) And,



**Problem 7.** Suppose we wanted to make an airplane invisible to a certain radar wavelength, say λ = 3cm. We could do this by coating the metallic surface of the plane with a polymer (for which say n = 1.50) to a certain thickness to ensure that the radar waves that reflect off the plane cancel each other out. Metallic surfaces have, for our purposes, an index of refraction of n = ∞ (they are really good reflectors of radar waves). So what minimum thickness would do the job? Be sure to do the whole derivation starting from Δφ = 2πm, m1/2.

The original wave (λ = 3cm) will be incident on the polymer, and will reflect, undergoing a phase shift. Transmitted wave will also under go a phase shift,



For the waves to cancel, we need:



And so the minimum d would be given by:



**Problem 8.** After having aced all of your exams you treat yourself to a bubble bath.  And you notice light shining through a thin soap bubble membrane with index of refraction n = 1.7 and thickness d = 0.3μm.  (a) What wavelengths of visible lighty would be most strongly transmitted through the membrane? Need to see wavelength derivations ☺. Is that getting old?

Hmmmmm……well the wavelengths that perfectly transmit are those which will destructively interfere upon reflection. When the wave hits the membrane it will reflect off the front surface and the back surface. It will invert off the front surface (call this the first reflected wave), but it won’t reflect off the back surface (call this the second reflected wave). So we have (it’s sooooo easy!):



Now λ2 is the wavelength in the bubble, not in air (and it’s *that* wavelength we need). So we say: n1λ1 = n2λ2 →



Filling in m­1/2 = 0.5, 1.5, 2.5, 3.5, we get λ = 510nm as the only wavelength in visible range.

(b) Which wavelengths would be most strongly reflected?

Now we want constructive interference upon reflection. So only thing that changes is the m:

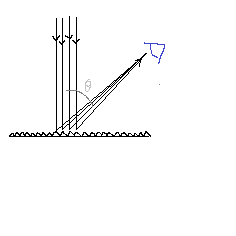
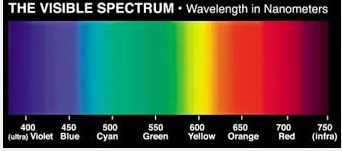


Now λ2 is the wavelength in the bubble, not in air (and it’s *that* wavelength we need). So we say: n1λ1 = n2λ2 →

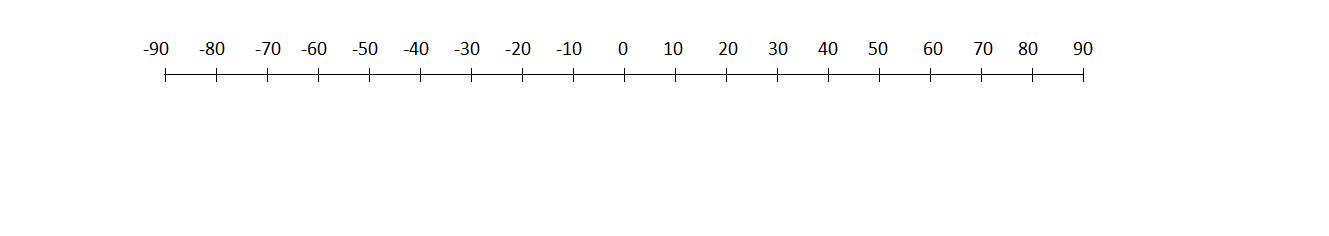


Filling in m­1/2 = 0, 1, 2, 3, we get λ = 680nm and 408nm as the wavelengthsin visible range.

**Problem 9.** Often times you’ll see a light spectrum coming from light reflecting off of a grated reflecting surface (like a CD or etched glass). Each groove acts like an independent source of light, and so the multiple grooves can illustrate multi-source interference. Suppose we have such a surface with grooves approximately 1.5 μm apart, and you’re standing a distance of 2m away. Let’s approximate white light as consisting of just three colors: red = 700nm, green = 550nm, and blue = 400nm light.

(a) Draw where you would expect to see the bright spots, for each color, at the angles delineated below. What would you see in the middle?



Angles to the bright spots, per wavelength are:



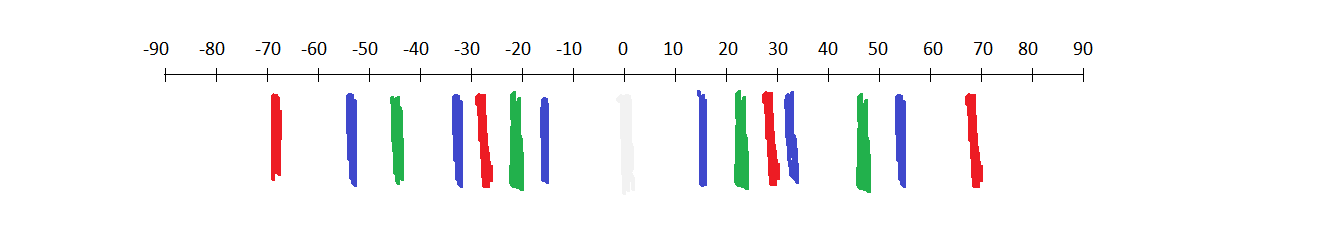
We get:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| blue | m = 0 | m = 1 | m = 2 | m = 3 | m = 4 |
| angle | 0 | 15 | 32 | 53 | dne |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| green | m = 0 | m = 1 | m = 2 | m = 3 | m = 4 |
| angle | 0 | 21 | 47 | dne | dne |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| red | m = 0 | m = 1 | m = 2 | m = 3 | m = 4 |
| angle | 0 | 28 | 69 | dne | dne |

So we get:



(b) What would decreasing the groove spacing do to the pattern?

Spread it out.

(c) What maximum groove spacing would suffice to eliminate all intensity maximums and minimums?

This would require a groove spacing which would send the first minimum to 90°. Furthermore, we’d need to send the blue minimum to 90°, as this would suffice to send all the others there too. So:



We need a spacing d < 0.2μm. You’d see white light basically, as all colors would be suffused throughout the screen.

(d) What would increasing the groove spacing do to the pattern?

It would bring them together.

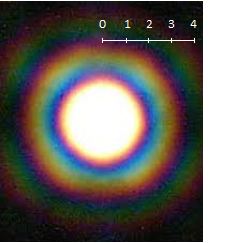
(e) What minimum groove spacing would would cause the distance between all visible maximums to be less than, say 1mm? You can use the small angle approximation sinθ ≈ tanθ.

The red wavelengths have the most spread apart spectrum. So if we make them less than 1mm, then all others will be too. So



You’d see white light again, basically because the variations in intensity would be too small for your eye to resolve.

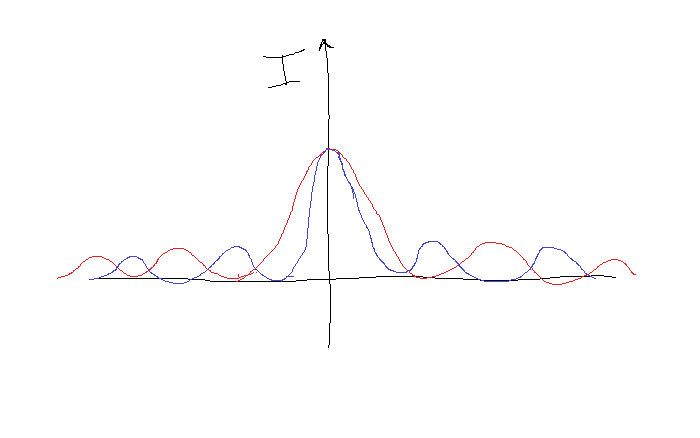
**Problem 10.** Light leaving a pinhole diffracts and produces the following pattern on the wall, 5m away. The scale is units of centimeters.



(a) Why is the center all white?

Because the zeroth order intensity maximums of all colors are at the center. And the superposition of all colors is white.

(b) Since blue has a smaller wavelength than red, its first maximum should have a smaller radius than red’s. But red appears at a smaller radius. By drawing the wiggly intensity pattern for both wavelengths, can you think of why you’re seeing red first?



You see red first because, expanding from the central maximum, it decays more slowly, and so will be more intense than the blue, until blue peaks again.

(c) Estimate the diameter of the pinhole.

We could use the radius to the first blue peak,



(d) What would happen to the pattern as you shrink the pinhole diameter?

It would expand outwards, i.e., all the radii would increase.

(e) What minimum diameter would suffice to eliminate all intensity maximums and minimums in visible rings? What color would you see?

So this would require the first blue minimum to go to infinity, i.e. 90°. So,



You would see white light, more or less as all colors overlap.

(f) What would happen as you increase the pinhole diameter?

The pattern would compress inwards. All the radii would shrink.

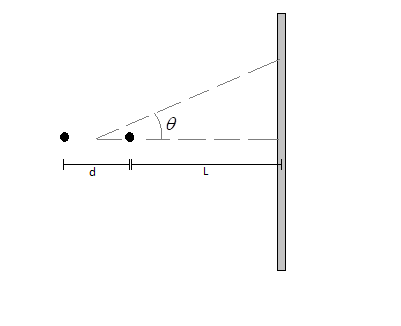
(g) Since the intensity of a circular diffraction pattern’s rings get smaller with radius, you can usually only see the first couple of rings. What diameter pinhole would compress the first 3 intensity maximums into a radius of say, 1mm, from the center? You can use small angle approximation sinθ ≈ tanθ. What would your eye see?

We’d have:



You’d see white again, because the variations in intensity would be too compressed for your eye to resolve.

**Problem 11.** Consider two point sources separated by a distance d, but perpendicular to the screen instead of parallel to it.



(a) What shape will the interference pattern take on the screen?

Concentric rings about the normal line.

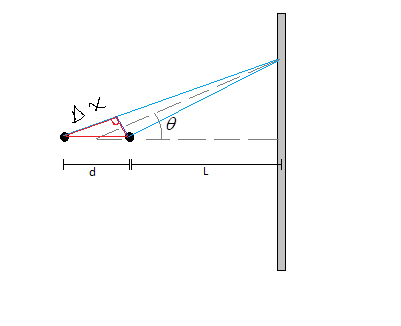
(b) As θ increases, will the path length difference between the sources become greater or smaller?

Smaller, unlike the other source arrangement we’ve been dealing with.

(c) Will there necessarily be constructive interference at 0°? What about at 90°?

Not at 0°. You could get constructive, or destructive, depending on what d is. But at 90°, you will get constructive interference because the path length difference will be close to 0.

(d) Derive a formula, using the small angle approximation (or large L approximation) for the angles to the interference maxima and minima. It’s similar to what we did in class.



So we have, for the maximums:



And similarly for minimums,



(e) Suppose λ = 1000nm, d = 2500nm. What will be the angle to the bright ring closest to the center (probably not your immediate impulse calculation)?

Well,



And the closest ring would be given by the smallest angle, which is given by m = 2.



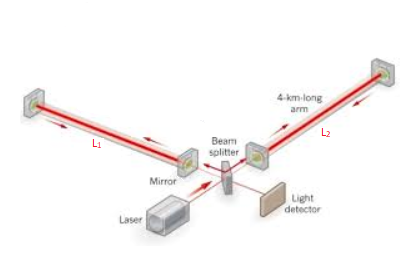
(f) How many bright rings will be visible?

We can just set the maximums on both sides equal,



So 2 bright rings will be visible. Note that unlike our parallel source situation, m = 0 corresponds to θ = 90 degrees, and larger m’s translate to *smaller* angles.

**Problem 12.** A simplified rendering of the LIGO detector (the thing which detects gravity waves) is this. A laser is aimed at a refracting surface (beam splitter) which partially reflects (ray 1) and partially transmits (ray 2) the ray towards mirrors at the end of arm L1 and L2, roughly 4km away in each direction (L1 and L2 are very close to the same length, but not quite). Rays 1 and 2 will reflect off the mirrors and head back towards the beam splitter. But LIGO uses addiotional mirrors which will reflect the rays back and forth down the arms, L1 and L2, 280 more times before they rejoin at the beam splitter. When they do finally rejoin at the beam splitter, it will pass and reflect rays 1 and 2 respectively towards the light detector. When they recombine at the light detector, they will create a circular diffraction pattern, just like the two point sources in the previous problem did. It’s easier to see why if you mentally rotate arm L2 onto arm L1. Then you can sort of see that you can view the ray traveling down arm L2 to be a distance d = 280(L2 – L1) = dΔL behind the ray traveling down arm L1.



(a) Ligo uses a laser with wavelength λ = 1064nm. What is the maximum number of maximum of bright rings you’ll see, for a given ΔL (see previous problem).



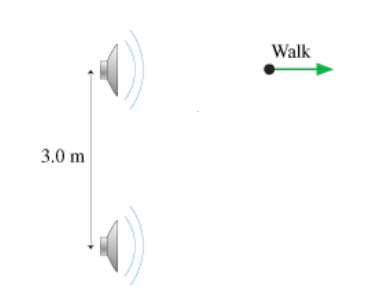
(b) Roughly, gravitational waves consist of contractions and rarefactions of space-time, much like sound waves consist of contractions and rarefactions of air molecules. But gravitational waves, by the time they reach us, are hopefully quite weak. So the stretch/compression of space would be small. Suppose a gravitational wave passes by and it presently expands L2 so that so that 5 more bright rings appear on the detector. By what approximate length did it expand one of the arms?

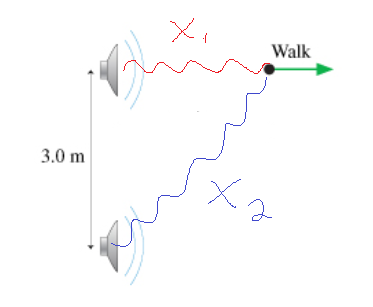
Pardon the awkward notation, but,



(c) This isn’t a question. So LIGO would appear to be able to measure lengths down to about 19/5 = 3.8nm. But it turns out that by using very sensitive photodetectors, LIGO, can measure much smaller expansions, down to 10-18m or so (about 1/1000 the size of a nucleus), by detecting even the *slightest* change in the interference pattern, even if it doesn’t result in a new bright ring appearing.

**Problem 13.**  You are standing directly in front of one of the two loudspeakers shown in the figure. They are 3.0 m apart and both are playing a 686 Hz tone. But speakers or light sources are not *always* in phase with each other. So let’s suppose the bottom speaker is exactly +π/4 out of phase with the top speaker.  As you begin to walk directly away from the speaker, at what distances from the speaker do you hear a *minimum* sound intensity? You can take the speed of sound to be 343m/s. Note you’ll have to do this *without* using small angle approximations. Also, you’ll need to use the more general expression for phase (the argument of the sign function) φ = kx – ωt + φ0, in order to properly account for the phase different between the sources.





So in order to get minimum interference, we would need



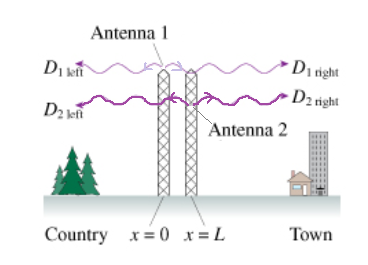
Now we would want to add the x1 to the other side and square both sides:



And now fill in λ = v/f = 343/686 = ½.



**Problem 14.** The broadcast antenna of an AM radio station is located at the edge of town. The station owners would like to beam all of the energy into town and none into the countryside, but a single antenna radiates energy equally in all directions, and so there is no way to ‘direct’ the energy with just one antenna. But we can solve the problem by introducing another antenna, which together with the first one would constitute what’s called a *phased array*. The figure shows two parallel antennas separated by distance L. Both antennas broadcast a signal at wavelength λ, but antenna 2 can delay its broadcast relative to antenna 1 by a time interval Δt in order to create a phase difference between the sources.



Suppose the station emits 1180kHz radio waves, described by the following equations.



What is the minimum L, and associated Δt that will give constructive interference of the waves on the right, and destructive on the left? As with the previous problem, you’ll need to use the general expression for phase, as the argument of the sin function. Also, remember the speed of EM waves is c = 3×108m/s.

So we need:



And,



Subtracting the top equation from the bottom gives us:



To get the smallest L we’d use m1/2 = ½, and m = 0. Then,



Adding the two equations together and using the same values of m1/2 and m, we can solve for Δt,

